SD 252 Signals & Systems December 9, 2004

Final Examination

Professor Paul Fieguth

Aids Permitted: *One* 8.5x11 page (both sides), *No* calculator.
Please turn off cellphones, pagers, and other electronic gadgets.

*** Well-drawn sketches / diagrams can be very helpful. ***

The grade value for each question is indicated in brackets [ ] next to the question number. I *will* give part marks for relevant statements or insights.

Manage your time!!! This is the most common problem which I see among students.

There are a lot of figures in this exam. See the last two pages for templates.

[2%] Bonus Question

Do **NOT** waste your time here unless you are happy with your answers to the rest of the exam!!!

To make it easier for me to find, an answer to the Bonus question must appear on the very back page of an exam booklet.

So, you have a hot cup of coffee. What is the fastest way to cool it?

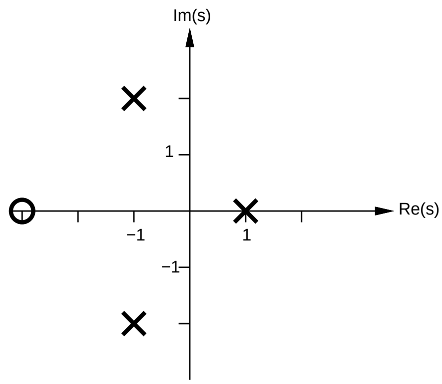
- First add some cream, stir, then use your breath to blow it.
- First blow it, stir, then add some cream.
- Both of the above end up with exactly the same temperature coffee.

Give a *convincing* argument or proof.

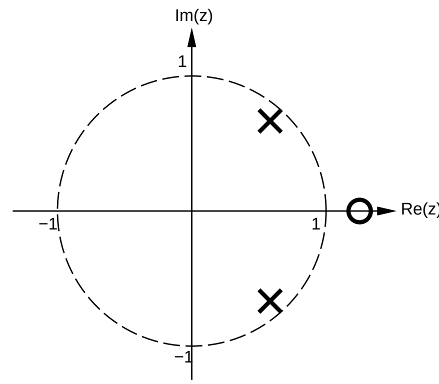
[32%] **1. Poles & Zeros**

*** **See last two pages of exam for figure templates.** ***

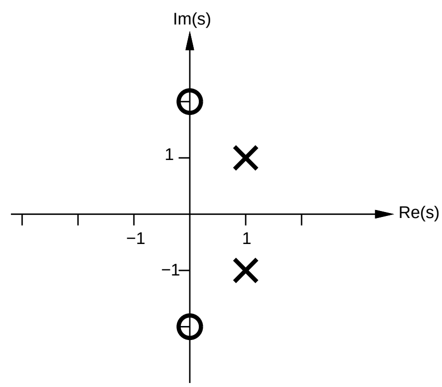
We want to examine different locations for poles and zeros. Consider the following four cases:



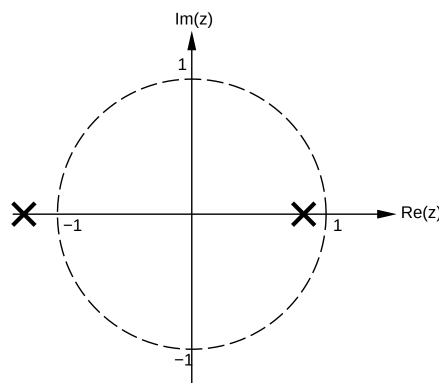
Case 1



Case 2



Case 3



Case 4

a) For *each* case, answer/sketch the following:

- Write down $H(s)$ or $H(z)$.
- Sketch the ROC assuming the associated system to be causal
- Sketch the ROC assuming the associated system to be stable
- Is the causal system stable? Why / why not?
- Does there exist a stable inverse to the system? Why / why not?

b) For Case 1, if the system is stable and has a D.C. gain of 6/5, find the impulse response $h(t)$ of the system. Possibly a useful Laplace transform:

$$\left\{e^{-at}cos(w_o t)u(t)\right\} \iff \frac{s+a}{(s+a)^2+w_o^2}$$

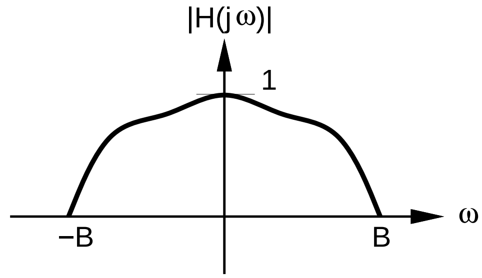
c) For Case 2, DC gain of 1.0, draw a sketch of the magnitude and phase of $H(z)$.

d) For Case 3, DC gain of 1.0, draw a sketch of the magnitude and phase of $H(s)$.

e) For Case 4, if the system is stable and has a D.C. gain of 10, find the impulse response $h(n)$ of the system.

[32%] **2. Sampling**

Suppose I give you an LTI system with the following frequency response magnitude:



Clearly the impulse response is band-limited.

a) Aside from LTI, there are four other properties which we may consider. Based on the above frequency response, briefly argue why each of invertibility, stability, causality, and memory must either be true, must not be true, or cannot be inferred from the figure.

b) Is it possible for this frequency response to be realized by a regular (i.e., linear, constant-coefficient) differential equation? Why / why not?

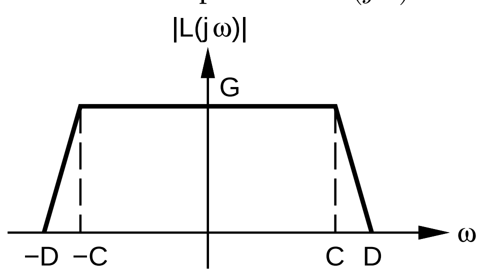
c) Suppose we sample the impulse response $h(t)$ associated with this system. As a function of B, what range of sampling rate ω_s allows, in principle, for a perfect recovery of $h(t)$ from $h(n)$?

d) Using the usual notation:

$$h_s(t) = \sum_k h(kT)\delta(t - kT) \qquad\qquad H_s(j\omega) = \mathcal{F}\{h_s(t)\}$$

where $T = 2\pi/\omega_s$, sketch the magnitude $|H_s(j\omega)|$, once for $\omega_s = 2B$, and once for $\omega_s = 4B$.

e) Suppose you are given a non-ideal low-pass filter $L(j\omega)$ as sketched:



That is, a flat gain of G to frequency C, and zero gain past D. The intent is to filter $h_s(t)$ through this non-ideal filter in order to perfectly recover $h(t)$.

As a function of B, what are the conditions on ω_s, C, D, G in order to have a perfect reconstruction?

f) (**Hard**)

Suppose that we have the non-ideal low-pass filter of (e), with $C = B, D = 2B$. This filter will clearly *not* reconstruct $h(t)$ if critically-sampled.

However, using ideas from modulation, show how this non-ideal low-pass filter *can* perfectly reconstruct $h(t)$ from $h_s(t)$, critically sampled, if we have available an ideal, perfect bandpass filter.

Sketch the frequency response magnitude of the bandpass filter which you need.

[16%] **3. Sampling of DEs**

In class we have seen how to sample signals in time, and the effect of sampling in the frequency domain. Here we want to see how sampling affects a differential equation.

Suppose we are given a system, whose input-output behaviour is characterized by the differential equation

$$y''(t) + y(t) = x'(t) + x(t)$$

a) Find $H(s)$, the transfer function associated with this DE.

b) Draw the pole-zero locations of $H(s)$.

We now have a rough understanding of the behaviour of the DE in continuous time. Suppose we sample with sampling period $T = 1$, thus

$$y(n) = y(t)\Big|_{t=n}$$

c) We will use a (poor) approximation of derivatives:

$$y'(t) \approx y(t) - y(t-1) \qquad\qquad y''(t) \approx y(t) - 2y(t-1) + y(t-2)$$

This is essentially the direct sampling of a differential equation to a difference equation. Write down the difference equation that results from this derivative approximation.

d) But does the result in (c) make any sense? From your difference equation of (c), find $H(z)$, and then plot the pole-zero locations. How do these pole-zero locations compare with those in part (b)?

e) What we have illustrated here is another approach to sampling: we can sample $H(s)$ to $H(z)$ by taking the poles and zeros of $H(s)$ and moving them to appropriate locations in the z-plane.

In general, given M zeros $(z_1, \ldots, z_M)$ and N poles $(p_1, \ldots, p_N)$ in the s-plane (continuous time), where should these poles and zeros be placed in the z-plane, given a sampling-rate of w_s?

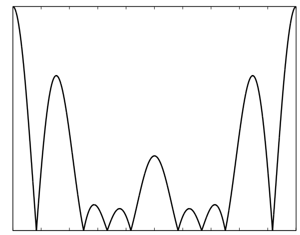
[20%] **4. Miscellaneous**

All of the following parts are completely independent of one another.

a) I have a signal $x(n)$ such that

$$x(n) = 0 \ \text{ for } n < 0 \text{ and for } n > 7,$$

$$x(n) \geq 0 \ \text{ for } 0 \leq n \leq 7.$$



The figure plots $|X(e^{j\omega})|$. Fully label the horizontal axis (what variable is being plotted, what are the values?). A figure template is available at the end of the exam.

Suppose I create a periodic signal from $x(n)$:

$$x_p(n) = x(n), 1 \leq n \leq 8 \ \text{ and } \ x_p(n+8) = x_p(n)$$

Plot the magnitude of the DTFS coefficients for x_p.

b) Suppose $h(t) = e^t u(t)$. For what values of s does the integral for $H(s)$ converge?

c) Sketch the magnitude and phase of

$$H(jw) = \frac{1}{(1-\omega) + j\sqrt{\omega}}$$

d) Given the Fourier transform pair

$$\mathcal{F}\left\{te^{-at}u(t)\right\} = \frac{1}{(a+j\omega)^2},$$

find the Fourier transform of

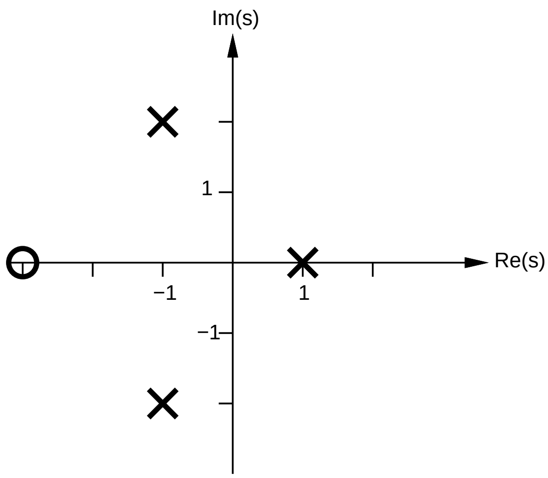
$$\frac{1}{(a+jt)^2}$$

e) Find the discrete-time Fourier transform of

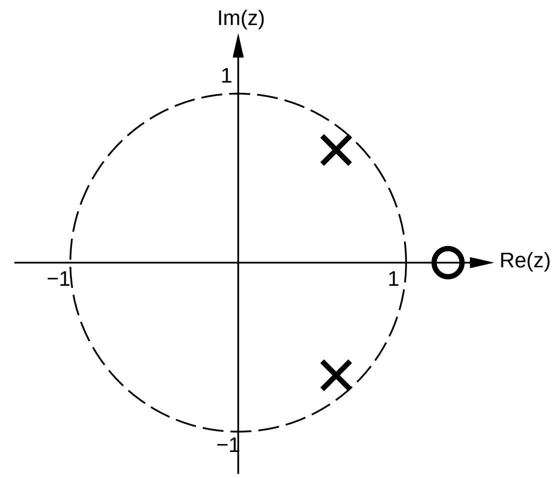
$$e^{3jn}\left(\frac{1}{2}\right)^n u(n)$$

You may find the following helpful for Questions 1, 4.
Make sure you clearly label what each plot is answering.

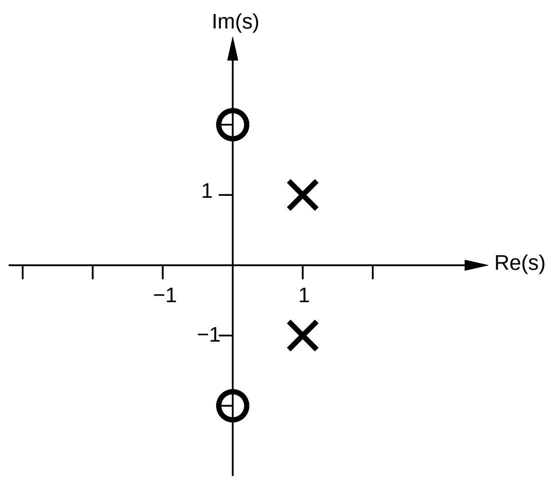
____
NAME



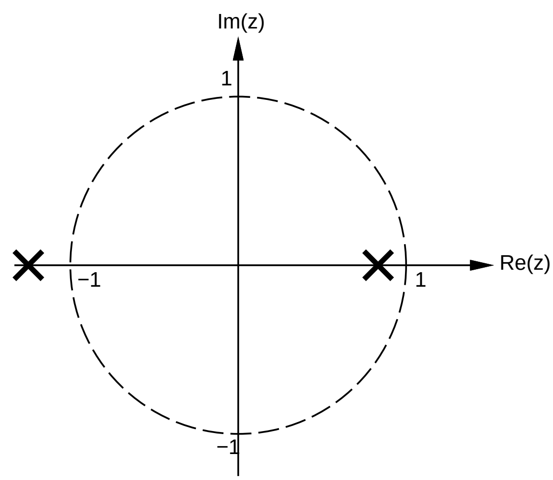
Case 1



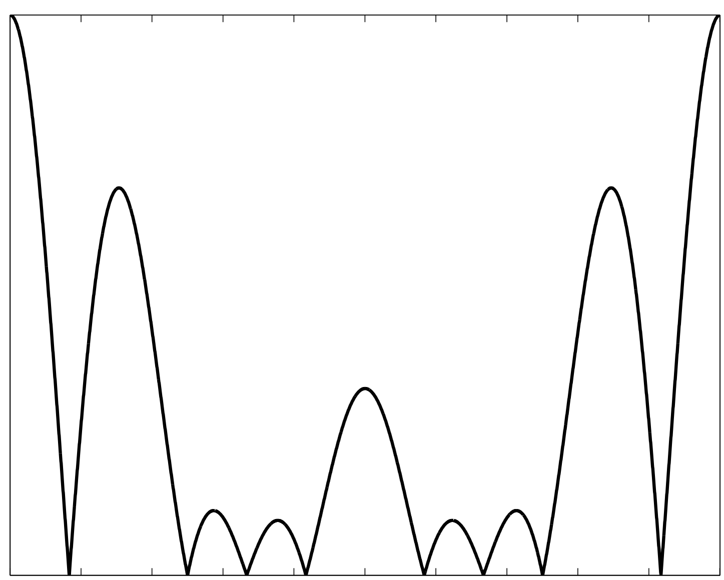
Case 2



Case 3



Case 4



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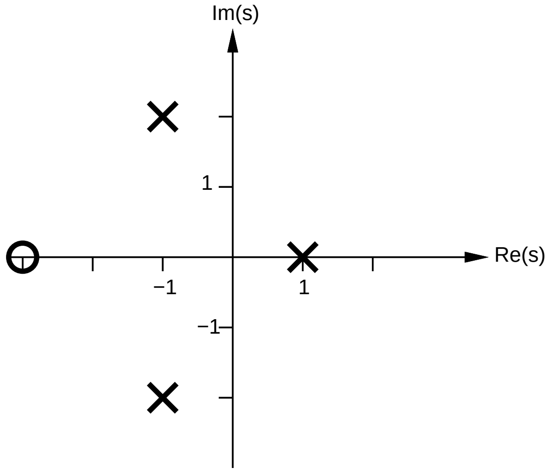
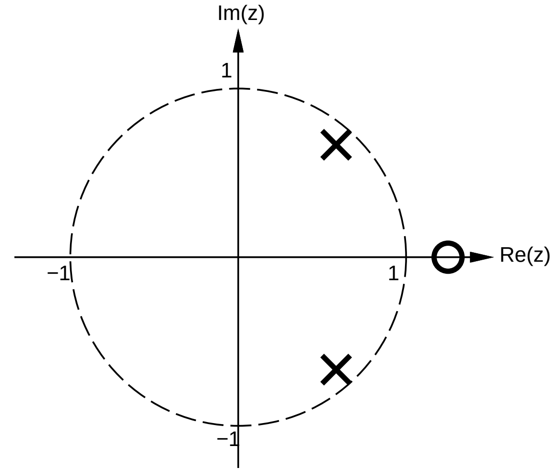
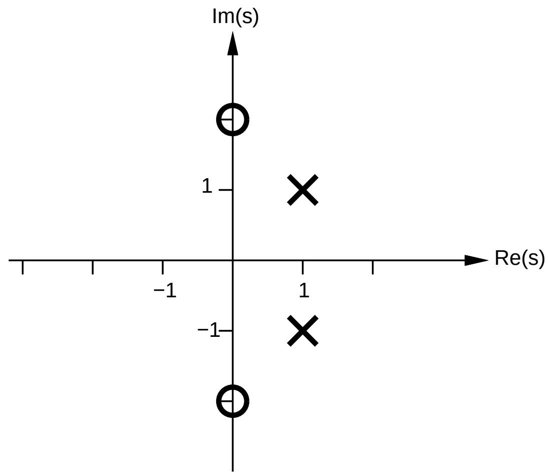


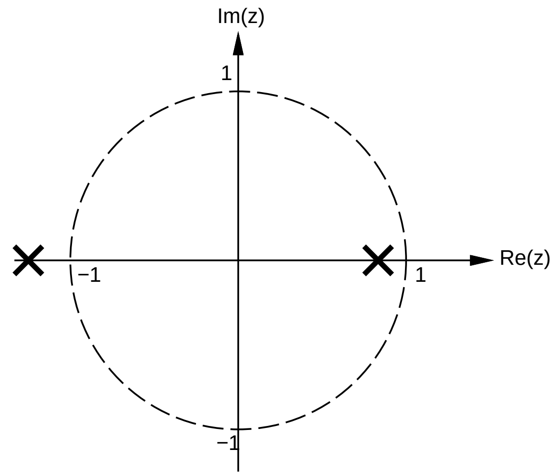
Case 1



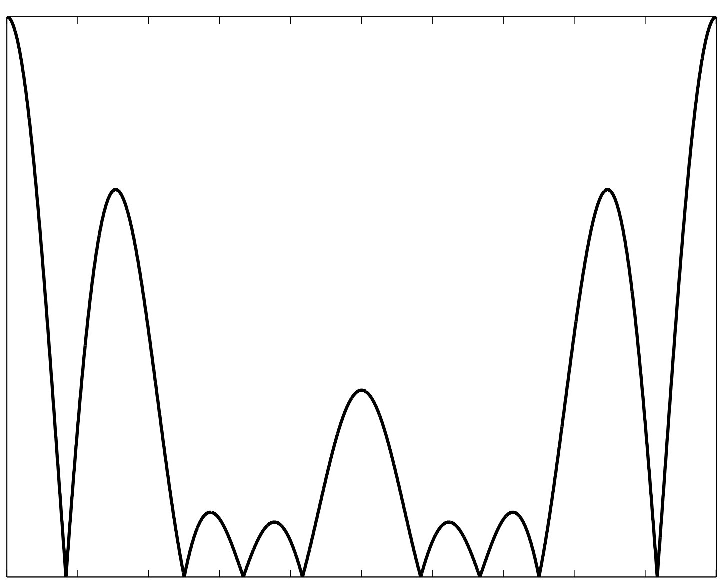
Case 2



Case 3



Case 4



PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

Property	Aperiodic Signal	Fourier Transform
	$x[n]$	$X(e^{j\omega})$ periodic with
	$y[n]$	$Y(e^{j\omega})$ period 2π
Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
Time Reversal	$x[-n]$	$X(e^{-j\omega})$
Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$
	$x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$j\mathcal{I}m\{X(e^{j\omega})\}$

Parseval's Relation for Aperiodic Signals

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

PROPERTIES OF THE FOURIER TRANSFORM

Property	Aperiodic signal	Fourier transform
	$x(t)$	$X(j\omega)$
	$y(t)$	$Y(j\omega)$
Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0}X(j\omega)$
Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega - \omega_0))$
Conjugation	$x^*(t)$	$X^*(-j\omega)$
Time Reversal	$x(-t)$	$X(-j\omega)$
Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}X(j\omega) * Y(j\omega)$
Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\} \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \\ \lvert X(j\omega)\rvert = \lvert X(-j\omega)\rvert \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ $[x(t)$ real$]$	$\mathcal{R}e\{X(j\omega)\}$
	$x_o(t) = \mathcal{O}d\{x(t)\}$ $[x(t)$ real$]$	$j\mathcal{I}m\{X(j\omega)\}$

Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$