Suppose we have $N$ independent samples $x_1, \ldots, x_N$ of a Gaussian random variable $x \sim N(\mu, \sigma^2)$. We want to estimate the mean, $\mu$.

1. Derive the ML estimator $\hat{\mu}_{ML}(x_1, \ldots, x_N)$.

2. Derive the estimation error variance $\sigma_e^2$ as a function of $N$ and $\sigma^2$.

3. Derive the CRB $\sigma_{CRB}^2$ for the error variance in estimating $\mu$. That is, find the bound on $\sigma_e^2$ as a function of $N$ and $\sigma^2$.

4. Now say that $\mu = 10, \sigma^2 = 10$. For $N = 1, 2, \ldots, 100$ simulate the above three steps and plot the results.

That is, for each value of $N$, generate random values $x_i$, calculate $\hat{\mu}$, and plot $\mu, \hat{\mu}, \mu \pm \sigma_{CRB}$ as sketched.

5. Next, if we didn’t actually know the value of $\sigma^2$, then we also don’t actually know the value of $\sigma_e$ and $\sigma_{CRB}$. So we’re motivated to estimate $\sigma_e$ from the data. So we can generate $N$ random values to find $\hat{\mu}$; then we can do the preceding $M$ times, yielding $\hat{\mu}_1, \ldots, \hat{\mu}_M$, from which we can calculate a sample variance.

Of course, you’re not actually getting $\sigma_e^2$ exactly; the sample variance $\hat{\sigma}_e^2$ is an estimate of $\sigma_e^2$.

So now I want you to derive the CRB for the error variance in estimating $\sigma_e$.

6. Again we want to simulate this for $N = 1, 2, \ldots, 100$, for two values of $M$: $M = 5, \ M = 50$. For each value of $M$, produce a plot superimposing the true $\sigma_e$, the estimated $\hat{\sigma}_e$, and $\sigma_e \pm \sigma_{e-CRB}$.

Give a brief interpretation / discussion of the results.